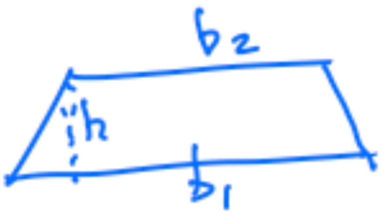
Geometric quantities

Area = $\frac{1}{2} b h$



Area = $\frac{1}{2}(b_1+b_2)h$

Area = πr^2

Circumference = $2\pi r$

Volume = xyz

Volume = Ah

Surface area

Ch

circumf. = C

$\pi r^2 h$ = Volume

Surface area

top = πr^2

bottom = πr^2

$2\pi r h$

volume $= \frac{1}{3}\pi r^2 h$

surface area

=

side view

$L = \sqrt{r^2+h^2}$

portion of circle

$$\pi L^2 \left(\frac{2\pi r}{2\pi L}\right)$$

$\Rightarrow$ Surface Area of

$\pi r L$

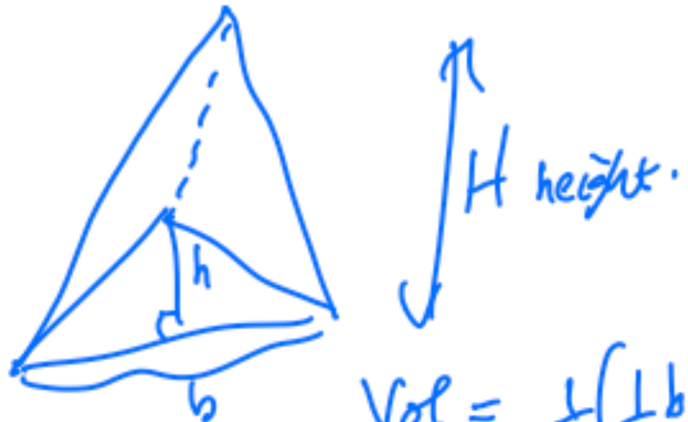
$= \pi r \sqrt{r^2+h^2}$

bottom area $= \pi r^2$

Generalization

Vol $= \frac{1}{3} A h$

eg

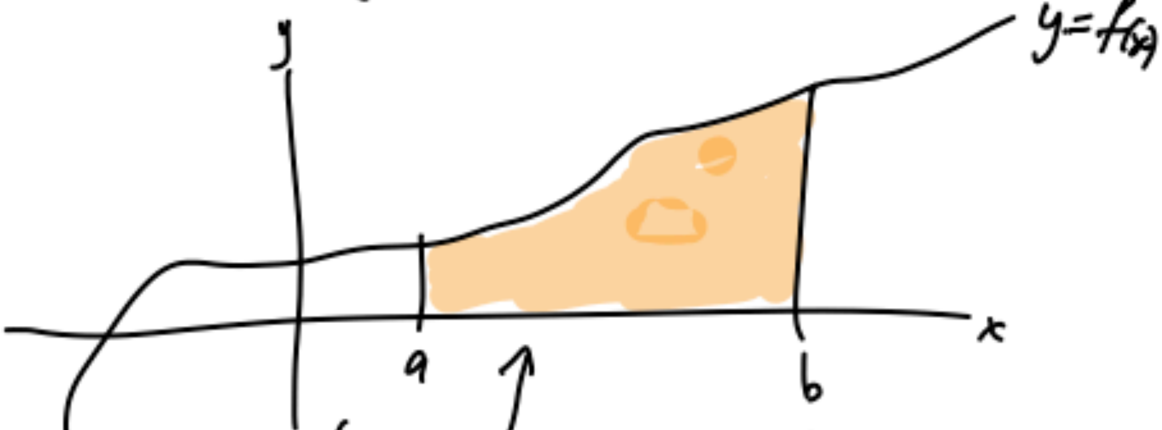


$$\text{Vol} = \frac{1}{3}\left(\frac{1}{2} b h\right) H$$

$$= \frac{1}{6} b h H$$

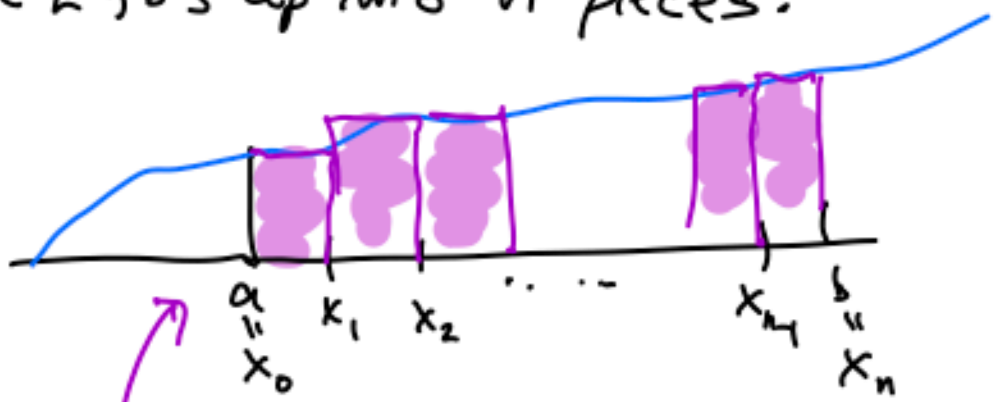
<u>New topic</u> — Integrals in higher dimensions!

Idea of a definite integral.



to compute: approximate by rectangles.

divide [a,b] up into n pieces.



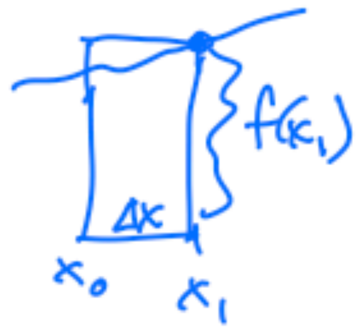
This area will give us an approximation. But if we take a limit as $n \to \infty$, we will get an exact area.

A lot of times, we will divide [a,b] into equal length segments, each of length $= \frac{b-a}{n}$.

What do we use for the height of each rectangle? — use $f(x)$ for some x in that interval.

To be definite, we'll choose the right hand side of each interval to make the height of each rectangle.

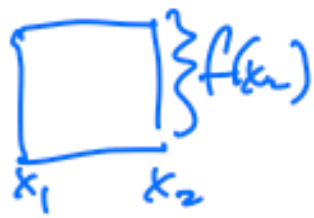
Then 1st rectangle



$$\text{Area} = f(x_1)\Delta x$$

2nd rectangle

$$\text{Area} = f(x_2)\Delta x$$



Total Area of rectangles:

$$f(x_1)\Delta x + f(x_2)\Delta x + \dots + f(x_n)\Delta x$$

$$= \sum_{j=1}^{n} f(x_j)\Delta x$$

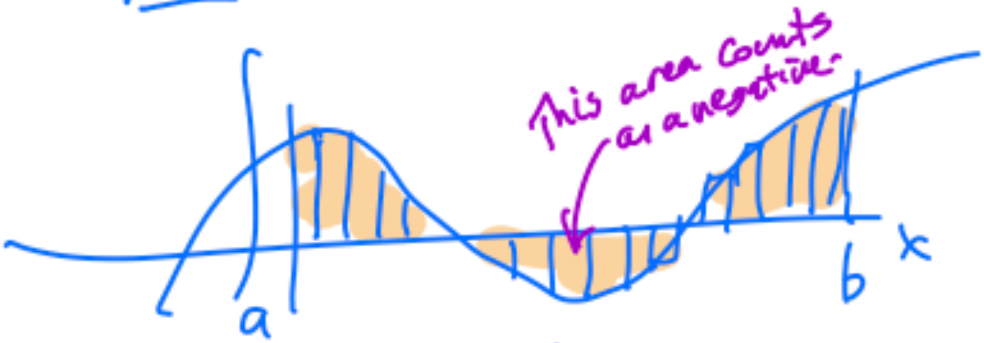
$$\Delta x = \frac{b-a}{n}$$
$$x_j = a + j\Delta x$$

$$x_0 = a$$
$$x_1 = a + \Delta x$$
$$x_2 = a + 2\Delta x$$
$$\vdots$$
$$x_j = a + j\Delta x$$

$$\text{exact area} = \underbrace{\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j)\Delta x}_{\text{definition of}} = \int_a^b f(x)\,dx.$$

Here: $\Delta x = \frac{b-a}{n}$, $x_j = a + j\Delta x$.

Note: If $f(x)$ is sometimes negative,



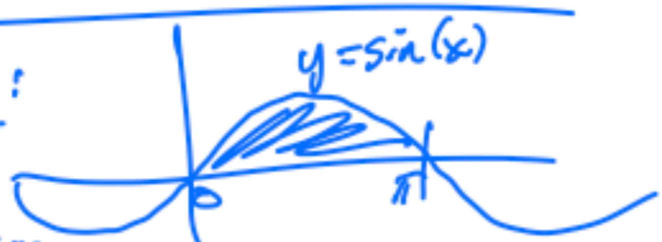
$\int_a^b f(x)\,dx =$ "signed area" under $y=f(x)$ between $x=a$ & $x=b$.

Big Theorem

Fundamental Theorem of Calculus.

$$\int_a^b \underbrace{F'(x)}_{\text{if } f(x)=F'(x)}\,dx = \underline{F(b)-F(a)}$$

Example:



What is this area?

Answer: $\int_0^{\pi} \sin(x)\,dx = \int_0^{\pi} (-\cos(x))'\,dx$

$$\overset{FTC}{=} \left(-\cos(x)\right)\Big|_{x=\pi} - \left(-\cos(x)\right)\Big|_{x=0}$$

$$= -(-1) - (-1) = \boxed{2}.$$

What is the sum version of this.

$$2 = \lim_{n\to\infty} \sum_{j=1}^{n} \sin(x_j)\Delta x$$

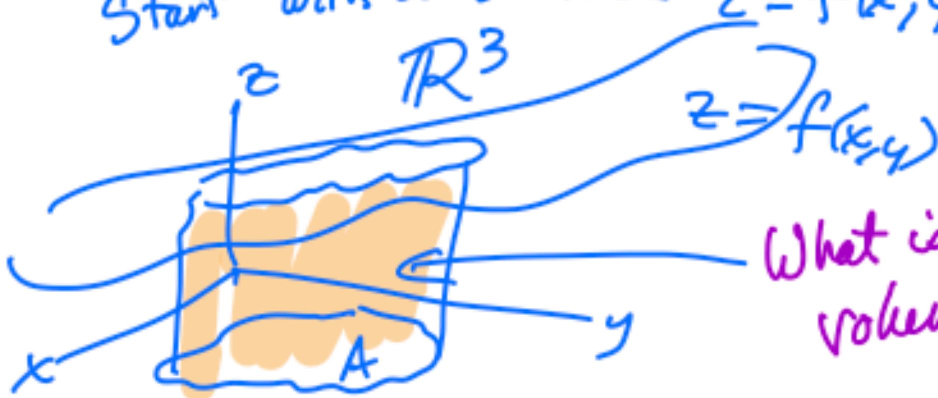
$$\Delta x = \frac{b-a}{n} = \frac{\pi - 0}{n} = \frac{\pi}{n}$$

$$x_j = a + j\Delta x$$
$$= 0 + j\left(\frac{\pi}{n}\right) = j\left(\frac{\pi}{n}\right)$$

$$2 = \lim_{n\to\infty} \sum_{j=1}^{n} \sin\left(\frac{j\pi}{n}\right)\frac{\pi}{n}.$$

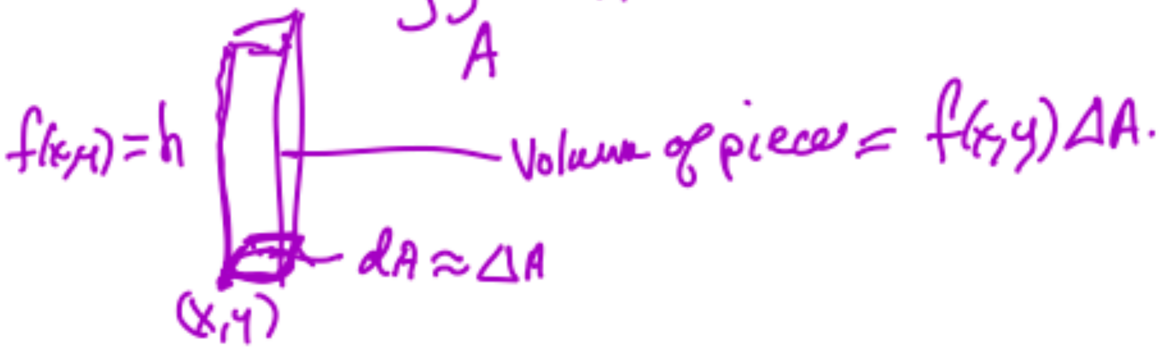
Generalization to Higher Dimensions.

Start with a surface $z = f(x,y)$ in $\mathbb{R}^3$



What is this volume?

Answer $\iint_A f(x,y)\, dA$



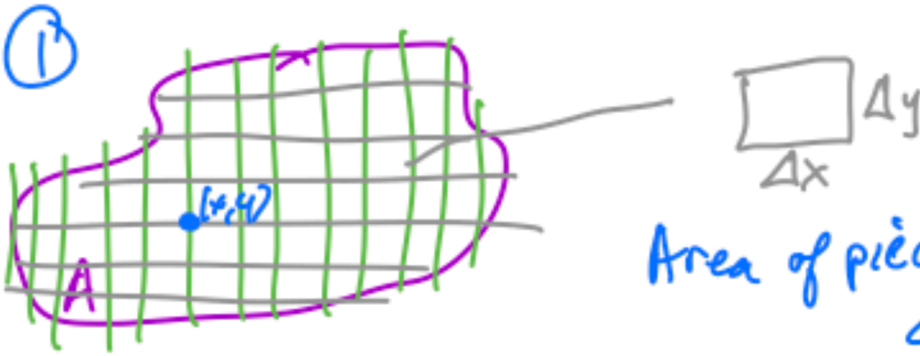
Plan: (1) Subdivide A area in xy plane into small pieces (rectangles)

(2) multiply $f(x,y)\cdot$(area of piece)

(3) add up.

(4) Take limit as # of pieces $\to \infty$.

(1)



Area of piece = $\Delta x \Delta y$

(2) $f(x,y)\Delta x \Delta y$ = volume between $z=f(x,y)$ & $z=0$, above that piece.

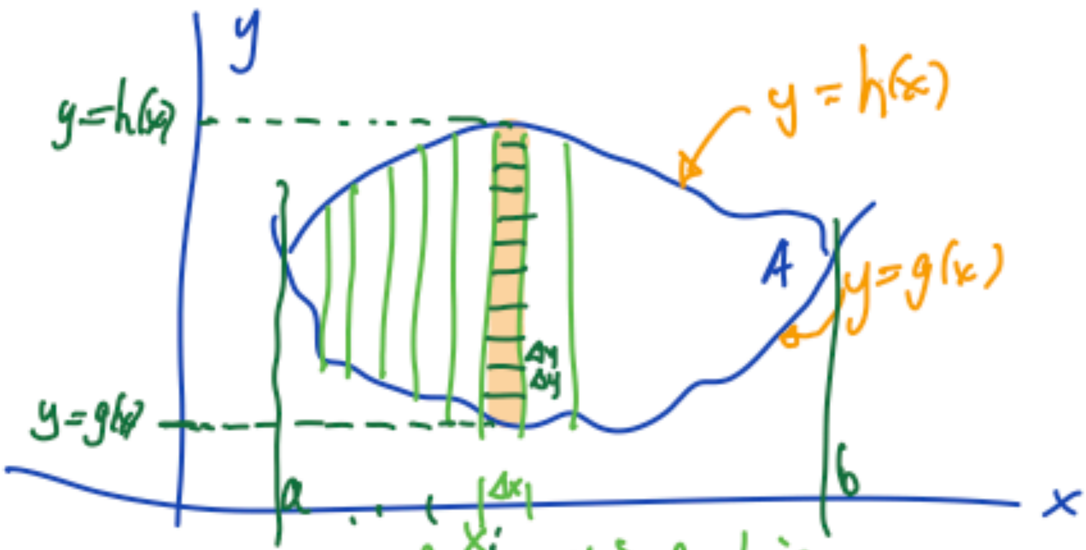
(3) $\sum f(x,y)\Delta x \Delta y$

(4) $\lim_{n\to\infty} \sum f(x,y)\Delta x \Delta y$ = integral.

Next question: How can we compute this?

$$\iint_A f(x,y)\,dx\,dy$$

How do we compute in an example?



First do vertical slices — compute just that part of the integral

Volume of $z=f(x,y)$ over this strip at x

$$\sum f(x,y_j)\Delta y\,\Delta x$$

$$\simeq \left(\int_{y=g(x)}^{y=h(x)} f(x,y)\,dy\right)\Delta x$$

1 variable integral where x is constant.

(only y is changing).

Whole volume = sum of what we just calculated over all the strips (at $x = x_j$)

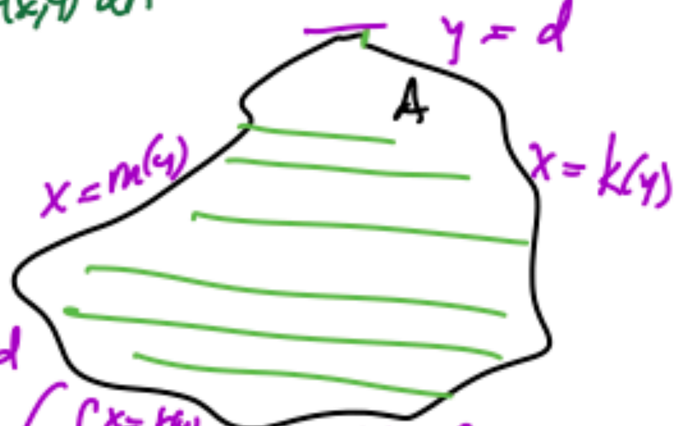
$$\begin{pmatrix}\text{Whole volume under} \\ z = f(x,y)\end{pmatrix} = \sum \int_{y=g(x)}^{y=h(x)} f(\underset{x_j}{x}, y)\, dy\, \Delta x$$

$$= \int_{x=a}^{x=b} \left(\int_{y=g(x)}^{y=h(x)} f(x,y)\, dy \right) dx$$

(vertical strip method)

$$\parallel$$

$$\iint_A f(x,y)\, dA$$

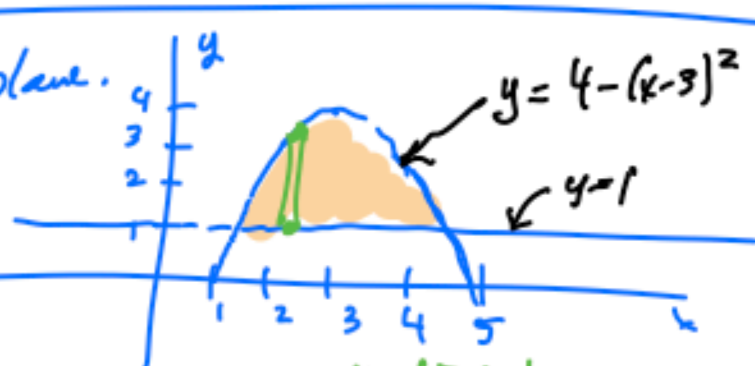


Alternative way

$$\text{Volume} = \iint_A f(x,y)\, dA = \int_{y=c}^{y=d} \left(\int_{x=m(y)}^{x=k(y)} f(x,y)\, dx \right) dy .$$

Example Find the volume under $z = xy + 20$ over the region in the xy plane between $y=1$ and $y = 4-(x-3)^2$.

Picture: xy plane.



vertical strip

$$\text{Volume} = \int_{x=}^{x=} \left(\int_{y=1}^{y=4-(x-3)^2} xy+20 \, dy \right) dx$$

intersection points: $4-(x-3)^2 = 1$

$$3 = (x-3)^2$$

$$\pm\sqrt{3} = x-3$$

$$x = 3 \pm \sqrt{3}.$$

$$\text{Volume} = \int_{x=3-\sqrt{3}}^{x=3+\sqrt{3}} \left(\int_{y=1}^{4-(x-3)^2} xy+20\, dy \right) dx$$

$$= \int_{x=3-\sqrt{3}}^{x=3+\sqrt{3}} \left(\left(\frac{xy^2}{2} + 20y \right) \Bigg|_1^{4-(x-3)^2} \right) dx$$

$$= \int_{3-\sqrt{3}}^{3+\sqrt{3}} \left[\frac{x}{2}\left(4-(x-3)^2\right)^2 + 20\left(4-(x-3)^2\right) - \left(\frac{x}{2}+20\right) \right] dx$$

$$= \underline{\text{answer}}$$